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New Numerical Method for Two-Dimensional Partially Wrinkled Membranes

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Based on the introduction of a nonnegative adjustable parameter into Poisson's ratio, a new numerical method is proposed to solve numerically the partially wrinkled membrane problems with arbitrary complex shapes. Three fundamental issues in this new method are addressed. First, a nonnegative adjustable parameter is introduced into the constitutive law, automatically representing unwrinkled (taut) and wrinkled states of a membrane. Second, a parametric variational principle is developed for the new membrane model. Third, by the variational principle, the original membrane problem is converted to a nonlinear mathematical programming problem. The variational principle and the mathematical programming formulation lay a foundation for efficient numerical analysis of wrinkled membrane structures using finite element discretization. Three typical examples are given to show the effectiveness of this method.

Nomenclature

$[B(x, y)]$ = element strain–displacement matrix
$[\tilde{D}]$ = modified constitutive matrix
E = Young's modulus
f, F = nonnegative constraint function
g = relation between the unknown displacements and the control parameters
h = thickness of membrane
$[K_i(\{\lambda\})]$ = stiffness matrices
$[N]$ = shape function matrix
NE = element number after finite element discretization
$\{p\}$ = boundary force vector corresponding to $\{\bar{u}\}$
$\{\bar{p}\}$ = vector of specified boundary force
$\{q\}$ = unknown global displacement vector
$\{q^e\}$ = nodal displacement vector
S_σ = specified force boundary
$\{\bar{T}\}$ = specified traction vector
u, v = x- and y-direction displacement components
$\{u\}$ = displacement vector
$\{\bar{u}\}$ = vector of specified boundary displacements
x, y = coordinate directions
$\Gamma(\{\lambda\})$ = nonnegative constraint function matrix
γ_{xy} = shear strain
δ = conventional variation operator
$\hat{\delta}$ = parametric variation operator
$\varepsilon_x, \varepsilon_y$ = x- and y-direction normal strains
$\{\varepsilon\}$ = plane strain vector
$\eta(\{\lambda\})$ = nonnegative constraint function vector
λ = nonnegative adjustable parameter
$\{\lambda\}$ = nonnegative adjustable parameter vector
μ = Poisson's ratio
$\tilde{\mu}$ = variable Poisson's ratio
Π = total potential energy
Π_λ = total potential energy with λ as the controlling parameter
σ_x, σ_y = x- and y-direction normal stresses
σ_1, σ_2 = first and second principle stresses
$\{\sigma\}$ = plane stress vector
τ_{xy} = shear stress
$\phi(a, b)$ = smoothing function for nonlinear complementarity problem
Ω = finite two-dimensional membrane region

Superscript

e = eth element

I. Introduction

THIN membranes stretched in tension are found in many technological applications. Examples are diverse, including fabric constructions, manufacture and handling of paper webs, films, textiles, metal foils and polymer sheets, and space inflatable structures such as radar antennas, solar arrays, and telescope reflectors. Membranes have little compression resistance and, therefore, are easy to wrinkle. Wrinkling significantly affects the quality, performance, and reliability of membrane structures and, thus, has been a research topic of continued interest.

In the literature, there are often two types of membrane models: 1) two-dimensional continua with zero flexural or bending stiffness[1,2] and 2) plate- or shell-like structures with small but non-negligible bending stiffness.[3] The current work is concerned with the first type and is focused on findings related to membranes with zero flexural stiffness.

Early works on membrane wrinkling date back to the 1920s and 1930s, when Wagner[4] and Reissner[5] developed a tension field theory, according to which, a membrane with zero flexural stiffness wrinkles when one in-plane principal stress is zero and the other is positive. Since then, many investigators have addressed different issues in the mechanics of wrinkled membranes. To measure partially wrinkled membranes, Stein and Hedgepeth[6] introduced the concept of variable Poisson's ratio, which was extended by Mikulas[7] in a numerical and experimental study and later by Miller and Hedgepeth[8] in a finite element implementation. With the introduction of a relaxed energy density, Pipkin[9] incorporated the tension field theory into the theory of elastic membranes, by which

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wrinkling is automatically represented. Steigmann and Pipkin[10] and Haseganu and Steigmann[11] further completed Pipkin's[9] theory. With the application of the concept of relaxed energy density, Jenkins and Leonard[12] conducted a finite element analysis of an inflated cylindrical cantilever and a submerged cylindrical membrane. Based on a fully nonlinear shell theory, Mote and Mockensturm[13] and Lin and Mote[14] investigated the wrinkles of thin, flat, and rectangular webs. Roddeman et al.[15] developed a method to evaluate the stresses in a wrinkled membrane. Kang and Im[1] proposed a numerical scheme for finite element analysis of wrinkled anisotropic membranes. Fujikake et al.[16] proposed a method of a shape-finding analysis to solve wrinkling problems of fabric tension structures. Recently, Yang et al.[17] proposed a bar-network membrane model for numerical solution of in-plane membrane wrinkling problems.

The current work is about numerical analysis of partially wrinkled membranes by use of an adjustable or variable material parameter. The concept of variable Poisson's ratio was first introduced by Stein and Hedgepeth[6] in obtaining closed-form solutions of wrinkled regions for a few simple-shaped thin membranes. In this paper, the Stein–Hedgepeth theory is generalized to develop a new numerical approach for analysis of wrinkled membranes with arbitrary shape and general boundary conditions. In the development of the approach, three fundamental issues have been addressed. First, through introduction of one variable parameter in a new constitutive law, unwrinkled and wrinkled states of membranes are systematically characterized with clear physical meaning. Second, a parametric variational principle for the new constitutive law is derived that facilitates a useful finite element formulation for wrinkling analysis of membranes of complex geometry and arbitrary boundary conditions. Third, the membrane problem is reduced to a nonlinear mathematical programming problem that can be solved by numerically efficient and accurate algorithms. This avoids commonly used iteration of membrane stresses, which often causes convergence problems in numerical wrinkling analyses.

The paper is arranged as follows. The wrinkling problem considered is described in Sec. II. The introduction of a nonnegative adjustable parameter into a membrane constitutive law is given in Sec. III. The parametric variational principle is derived in Sec. IV. Nonlinear mathematical programming procedure for the membrane model is presented in Sec. V. A parametric finite element formulation for wrinkling analysis is formed in Sec. VI. To show the effectiveness of the proposed approach, three numerical examples are provided in Sec. VII.

II. Statement of Problem

In this study, the following assumptions about membranes are made: Membranes have zero flexural stiffness and cannot carry compression stresses, membranes are composed of isotropic elastic materials, membranes are flat before deformation and undergo in-plane deformations in a state of plane stress, and no slack regions appear in wrinkled membranes. Under these assumptions, a partially wrinkled membrane confined in a finite two-dimensional domain is governed by the equilibrium equations[18]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \tag{1}$$

and the strain–displacement relation

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2}$$

along with appropriate boundary conditions. In a wrinkling analysis, zero body forces are usually assumed, and external tension forces are specified at the membrane boundary.

The conventional stress–strain relation in two-dimensional elasticity theory is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\mu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{3}$$

This relation allows negative stresses to appear, which is invalid for a wrinkled membrane because it cannot carry compressive stresses. Thus a different constitutive law describing the mechanics of wrinkling is required. In accord with the tension field theory[4] and stress criteria,[2] wrinkles eliminate compressive stresses completely and, hence, only allow nonnegative principal stresses in the membrane. In general, a particle point of the membrane is in one the following three states defined by principal stresses.[19,20]

Taut state (S1):

$$\sigma_2 > 0 \tag{4a}$$

Wrinkled state (S2):

$$\sigma_1 > 0, \qquad \sigma_2 \le 0 \tag{4b}$$

Slack state (S3):

$$\sigma_1 \le 0 \tag{4c}$$

In the Stein–Hedgepeth theory,[6] a wrinkled membrane can be divided into two types of regions, wrinkled and unwrinkled (or taut). In a taut region, the membrane is linear elastic and Eq. (3) applies. In a wrinkled region, the behavior of the membrane is described by the following modified constitutive law:

$$\{\sigma\} = [\tilde{D}]\{\varepsilon\} \tag{5}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$[\tilde{D}] = \frac{E}{1-\tilde{\mu}^2} \begin{bmatrix} 1 & \tilde{\mu} & 0 \\ \tilde{\mu} & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\tilde{\mu}) \end{bmatrix} \tag{6}$$

and $\tilde{\mu}$ is the variable Poisson's ratio. Equation (3) is a special case of Eq. (5) when $\tilde{\mu}$ is replaced by the regular Poisson's ratio.

According to the previous discussion, a fundamental problem in modeling and analysis of two-dimensional wrinkled membranes (without slack regions) is to solve the boundary-value problem governed by Eqs. (1) and (2) and a nonlinear constitutive relation characterizing two membrane states by expressions (4a), (4b), (5), and (6). The Stein–Hedgepeth theory[6] was applied to produce closed-form solutions for a few simple-shaped membranes. In the subsequent sections, a new numerical method is developed based on this theory, which can deal with wrinkled membrane of arbitrary shape and boundary conditions.

III. One-Variable-Parameter Membrane Model

To solve the boundary-value problem governed by Eqs. (1) and (2) and the nonlinear membrane constitutive relation, introduce a nonnegative adjustable parameter λ into the variable Poisson's ratio as

$$\tilde{\mu} = \mu + \lambda \tag{7}$$

where μ is the regular Poisson's ratio (a constant). The parameter λ is adjusted such that the conditions (4a) and (4b) of the membrane states are satisfied. It can be shown that λ meets the following conditions for the taut and wrinkled membrane states.

Taut state (S1):

$$\lambda = 0 \tag{8a}$$

Wrinkled state (S2):

$$\lambda > 0 \tag{8b}$$

The parameter λ physically represents the relaxation of the second principal stress when the membrane is in a wrinkled state. For a wrinkled state, a positive λ renders $\tilde{\mu} > \mu$, which in turn reduces the principal σ_2 or sets it to zero. For a taut state, no stress relaxation is needed and, therefore, λ becomes zero.

By Eqs. (4a), (4b), and (8), the following complementary equation is true for both the taut and wrinkled membrane states:

$$\sigma_2 \cdot \lambda = 0 \tag{9}$$

The stress criteria for the taut and wrinkled states of a membrane can be stated as follows.

Taut state (S1):

$$\sigma_2 \geq 0, \qquad \lambda = 0 \tag{10a}$$

Wrinkled state (S2):

$$\sigma_2 = 0, \qquad \lambda > 0 \tag{10b}$$

In summary, the governing equations of two-dimensional wrinkled membranes with no slack state are the equilibrium equations (1), the strain–displacement relation described in Eq. (2), and the constitutive relationship given by Eqs. (5), (7), (9), and (10). For convenience, we shall call the membrane governed by those equations the one-variable-parameter (1-VP) membrane model.

IV. Parametric Variational Principle

Whereas the 1-VP membrane model given in Sec. III is capable of characterizing taut and wrinkled states of membranes, it does not automatically provide an approach to the solution of the membrane problem. In this and next sections, a parametric variational principle and a nonlinear complementarity problem (NCP) of mathematical programming are derived, which make it possible to obtain accurate numerical solutions of the considered wrinkling problem.

For a partially wrinkled membrane confined in a finite two-dimensional region Ω, its total potential energy is[18]

$$\Pi(\{u\}, \lambda) = \int_\Omega \frac{1}{2}\{\varepsilon\}^T[\tilde{D}(\lambda)]\{\varepsilon\}h \, \mathrm{d}\Omega - \int_{S_\sigma} \{\bar{T}\}^T\{u\} \, \mathrm{d}S \tag{11}$$

Although the minimum potential energy principle (MPEP) has been widely used in many static problems of solids and structures, it is not directly applicable to the current membrane problem. In a conventional sense, the parameters λ contained in the matrix $[\tilde{D}]$ must participate in the variation of Π because they are related to membrane strains. This would eventually lead to erroneous Euler equations that differ from the equilibrium equations (1). In addition, the stress–strain relation for the wrinkled membranes, described by the inequality conditions (5), (9), and (10), cannot be handled by conventional variational principles such as the MPEP.

To overcome the mentioned difficulties, the parametric variational principle (PVP)[21,22] is adopted. One key in the PVP is to introduce certain control parameters, by which the inequality conditions that describe the nonlinear constitutive law of a wrinkled membrane are explicitly expressed as a new type of constraint conditions. These constraint conditions are imposed on the energy functional of the membrane, instead of being directly embedded in the functional as in the MPEP. As a result, parametric variation, that is, functional variation with the control parameters being treated as constants, yields correct equilibrium equations. Unlike Lagrange multipliers, the control parameters do not participate in functional variation.

For the membrane problem in consideration, the variable parameter λ defined in Sec. III is chosen as a control parameter. The potential energy in Eq. (11), thus, becomes

$$\Pi_\lambda(\{u\}) = \int_\Omega \frac{1}{2}\{\varepsilon\}^T[\tilde{D}]\{\varepsilon\}h \, \mathrm{d}\Omega - \int_{S_\sigma} \{\bar{T}\}^T\{u\} \, \mathrm{d}S \tag{12}$$

where the displacement vector $\{u\} = \{u, v\}^T$. Although the expression in Eq. (12) is the same as that in Eq. (11), $\Pi_\lambda(\{u\})$ indicates that the displacements $\{u\}$ are the only independent arguments of the functional and that λ contained in $[\tilde{D}]$ is not involved in the variation. Furthermore, by Eqs. (2) and (5), the second principle stress can be expressed in terms of the membrane displacements and control parameters:

$$\sigma_2 = f(\{u\}, \lambda) \tag{13}$$

The function f, which is called a constraint function, is nonnegative by conditions (10). For the purpose of later developments, the constitutive law of the wrinkled membrane is restated as follows.

Claim 1. The constitutive law of the 1-VP model given in Sec. III is

$$\{\sigma\} = [\tilde{D}]\{\varepsilon\} \tag{14a}$$

which is subject to the conditions

$$f(\{u\}, \lambda) \cdot \lambda = 0, \qquad \lambda \geq 0, \qquad f(\{u\}, \lambda) \geq 0 \tag{14b}$$

Let $\hat{\delta}$ be the parametric variation operator, which acts on $\{u\}$, but not on λ. The first parametric variation of Π_λ is

$$\hat{\delta}\Pi_\lambda = \int_\Omega \{\varepsilon\}^T[\tilde{D}]\delta\{\varepsilon\}h \, \mathrm{d}\Omega - \int_{S_\sigma} \{\bar{T}\}^T \delta\{u\} \, \mathrm{d}S \tag{15}$$

Because $[\tilde{D}]$ is treated as a constant matrix in the parametric variation, $\hat{\delta}\Pi_\lambda$ is similar in format to the conventional variation of the potential energy of a classical two-dimensional elasticity problem. Therefore, it is easy to show that the Euler equations of $\hat{\delta}\Pi_\lambda = 0$ are the original equilibrium equations (1).[22]

V. NCP for Wrinkled Membranes

The PVP given in Sec. IV allows one to develop new solution methods for the membrane problem. One idea is to convert the original membrane problem to an NCP that can be accurately and efficiently solved, without the need for stress iteration. To this end, consider the variation $\hat{\delta}\Pi_\lambda = 0$, from which the membrane displacements can be viewed as a function of the control parameter, namely,

$$\{u\} = g(\lambda) \tag{16}$$

It follows that the second principal stress can also be expressed by the control parameters

$$\sigma_2 = F(\lambda) \equiv f(g(\lambda), \lambda) \tag{17}$$

where Eq. (13) has been used. Hence, with the constraint conditions (14b), the original membrane problem is reduced to an equivalent NCP of the mathematical programming problem as follows.

Claim 2. The solution of the problem of the wrinkled membrane described in Secs. II and III is equivalent to that of the following NCP:

Find λ, such that

$$F(\lambda) \cdot \lambda = 0 \tag{18a}$$

subject to the conditions

$$\lambda \geq 0, \qquad F(\lambda) \geq 0 \tag{18b}$$

In the preceding NCP, the control parameter λ is the only unknown to be determined. The NCP can be numerically solved by a smoothing Newton method (see Refs. 23 and 24). In an NCP-based analysis, the membrane stress–strain relation does not have to be satisfied during the solution process; it will eventually met when the final solution λ of the NCP is obtained. This way, initial guesses, iteration of stresses, and related convergence problems that are encountered in conventional wrinkling analyses can be completely avoided.

VI. Finite Element Discretization

In this section, a finite element formulation is developed based on the 1-VP membrane model, the PVP, and the NCP given in Sec. V.

Divide the region Ω of a membrane of arbitrary shape and boundary conditions into *NE* elements. The total potential energy of the membrane, by Eq. (12), can be written as[25]

$$\Pi_\lambda = \sum_{e=1}^{NE} \Pi_\lambda^{(e)} \tag{19}$$

where

$$\Pi_\lambda^{(e)} = \int_{\Omega^e} \frac{1}{2}\{\varepsilon^e\}^T[\tilde{D}(\lambda^e)]\{\varepsilon^e\}h\,\mathrm{d}\Omega - \int_{S_\sigma^{(e)}} \{\bar{T}\}^T\{u^e\}\,\mathrm{d}S \qquad (20)$$

Following standard finite element interpolation, the displacements, strains, and stresses of the eth element Ω^e are expressed as

$$\{u^e(x, y)\} = [N(x, y)]\{q^e\}, \qquad (x, y) \in \Omega^e \qquad (21a)$$

$$\{\varepsilon^e\} = [B(x, y)]\{q^e\} \qquad (21b)$$

$$\{\sigma^e\} = [\tilde{D}(\lambda^e)][B(x, y)]\{q^e\} \qquad (21c)$$

where the vector $\{q^e\}$ contains unknown nodal displacements of the element. Note that the element has a control parameter λ^e, as shown in Eq. (21c). Substitute Eqs. (21) into Eq. (19) to obtain

$$\Pi_\lambda = \tfrac{1}{2}\{q\}^T[K_0(\{\lambda\})]\{q\} + \{q\}^T[K_1(\{\lambda\})]\{\bar{u}\} + \tfrac{1}{2}\{\bar{u}\}^T[K_2(\{\lambda\})]\{\bar{u}\} - \{q\}^T\{\bar{p}\} - \{\bar{u}\}^T\{p\} \qquad (22)$$

where the vector $\{\lambda\}$ consists of the control parameters of all elements, namely,

$$\{\lambda\} = \{\lambda^1 \ldots \lambda^{NE}\}^T \qquad (23)$$

and the global displacement vector $\{q\}$ is formed from the elements of $\{q^e\}$. The matrices $[K_i(\{\lambda\})]$, $i = 0, 1, 2$, are obtained from the first integral in Eq. (20).

According to the PVP in Sec. IV, the first parametric variation is $\partial\Pi_\lambda/\partial\{q\} = 0$, which leads to

$$[K_0(\{\lambda\})]\{q\} + [K_1(\{\lambda\})]\{\bar{u}\} - \{\bar{p}\} = 0 \qquad (24)$$

Rearrange Eq. (24) to express the global displacement vector as a function of the control parameter, that is,

$$\{q\} = g(\{\lambda\}) \qquad (25)$$

with

$$g(\{\lambda\}) = -[K_0(\{\lambda\})]^{-1}([K_1(\{\lambda\})]\{\bar{u}\} - \{\bar{p}\}) \qquad (26)$$

Note that $\{q\}$ is only dependent on the control parameters $\{\lambda\}$ because $\{\bar{u}\}$ and $\{\bar{p}\}$ are specified boundary displacements and force parameters.

In Sec. V, the original membrane problem is converted to a NCP. According to Claim 1, with finite element discretization, the stress–strain relation of the eth element is subject to the conditions

$$f^e(\{u^e\}, \lambda^e) \cdot \lambda^e = 0 \qquad (27a)$$

$$\lambda^e \geq 0, \qquad f^e(\{u^e\}, \lambda^e) \geq 0 \qquad (27b)$$

For a nonconstant stress finite element, f^e represents the averaged second principal stress of the element.

To formulate a nonlinear complementarity problem, the nodal displacement vector of the eth element, by Eq. (25), is expressed as

$$\{q^e\} = g^e(\{\lambda\}) \qquad (28)$$

Substitute Eq. (28) into Eq. (27a) to define the following new functions of $\{\lambda\}$:

$$\boldsymbol{\eta}^e(\{\lambda\}) \equiv f^e(g^e, \{\lambda^e\}) \qquad (29)$$

Now, replace f^e in Eq. (27) by $\boldsymbol{\eta}^e$ in Eq. (29) and group the resulting equations and inequalities to yield

$$\Gamma(\{\lambda\}) \cdot \{\lambda\} = 0 \qquad (30a)$$

$$\{\lambda\} \geq 0, \qquad \boldsymbol{\eta}(\{\lambda\}) \geq 0 \qquad (30b)$$

where the matrix Γ and vector $\boldsymbol{\eta}$ are of the form

$$\Gamma(\{\lambda\}) = \mathrm{diag}(\boldsymbol{\eta}^1 \ldots \boldsymbol{\eta}^{NE}), \qquad \boldsymbol{\eta}(\{\lambda\}) = \{\boldsymbol{\eta}^1 \ldots \boldsymbol{\eta}^{NE}\}^T \qquad (31)$$

In mathematical programming theory, expressions (30a) and (30b) form an NCP. This NCP can be solved by an efficient smoothing Newton method, which is recently developed and completed (see Refs. 25–27). The current wrinkling analysis makes use of the Chen–Harker–Kanzow–Smale smoothing function (see Refs. 26 and 28):

$$\phi(a, b) = \left(\sqrt{a^2 + 4b^2} + a\right)/2, \qquad -\infty < a < +\infty, \quad b > 0 \qquad (32)$$

VII. Numerical Examples

The 1-VP membrane model with the corresponding numerical method is illustrated in the wrinkling analysis of three membranes: a triangular membrane, a bending stretched rectangular membrane, and an annular membrane. The finite element formulation for the 1-VP membrane model uses constant strain triangular elements. For the triangular membrane, the numerical results obtained by the proposed method are compared with those obtained by the nonlinear finite element analysis software ABAQUS.

Example 1: Triangular Membrane

Figure 1 shows an isosceles triangular membrane with long edge of 40 m and angle $\theta = 45$ deg. The three corners of the membrane are cut flat to reduce the effects of the concentrated stresses. Corner A is fixed, whereas two other corners are stretched by the known two identical tension forces $P = 3.45$ N in angle θ_p. The thickness and material parameters of the membrane are $h = 5.0\,\mu$m, $E = 2.172$ GPa, and $\mu = 0.3$, respectively.

Let $\theta_p = 22.5$ deg. The wrinkle patterns are predicted by the proposed numerical method. In the numerical simulation, four different meshes of 54, 204, 400, and 800 constant triangular elements, respectively, are used in finite element discretization. Figures 2a–2d show the wrinkled region of the membrane, where a wrinkled element is marked by the dumbbell-like symbol that expresses the magnitude and orientation of the first principle stress at the center of the wrinkled element. All of these dumbbells form a wrinkled region or pattern of the membrane. As can be seen, the wrinkled region is convergent with the increase of the number of elements. The wrinkled region given in Fig. 2d is in good agreement with the result obtained by software ABAQUS with a mesh of 100 triangular elements and 4950 quadrant elements (Figs. 3a and 3b).

Listed in Table 1 is the normal stress σ_x at point P1 ($x = 4$ m and $y = 4$ m) and point P2 ($x = 12$ m and $y = 4$ m), which is computed by

Table 1 Normal stress of the triangular membrane

Finite element mesh number of elements	σ_x, MPa, at point P1[a]	σ_x, MPa, at point P2[b]
54	3.106	5.119
204	3.190	5.138
400	0.428	0.663
800	0.432	0.660

[a]Here $x = 4$ m and $y = 4$ m. [b]Here $x = 12$ m and $y = 4$ m.

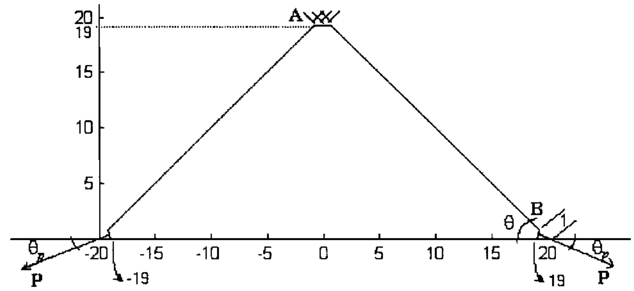
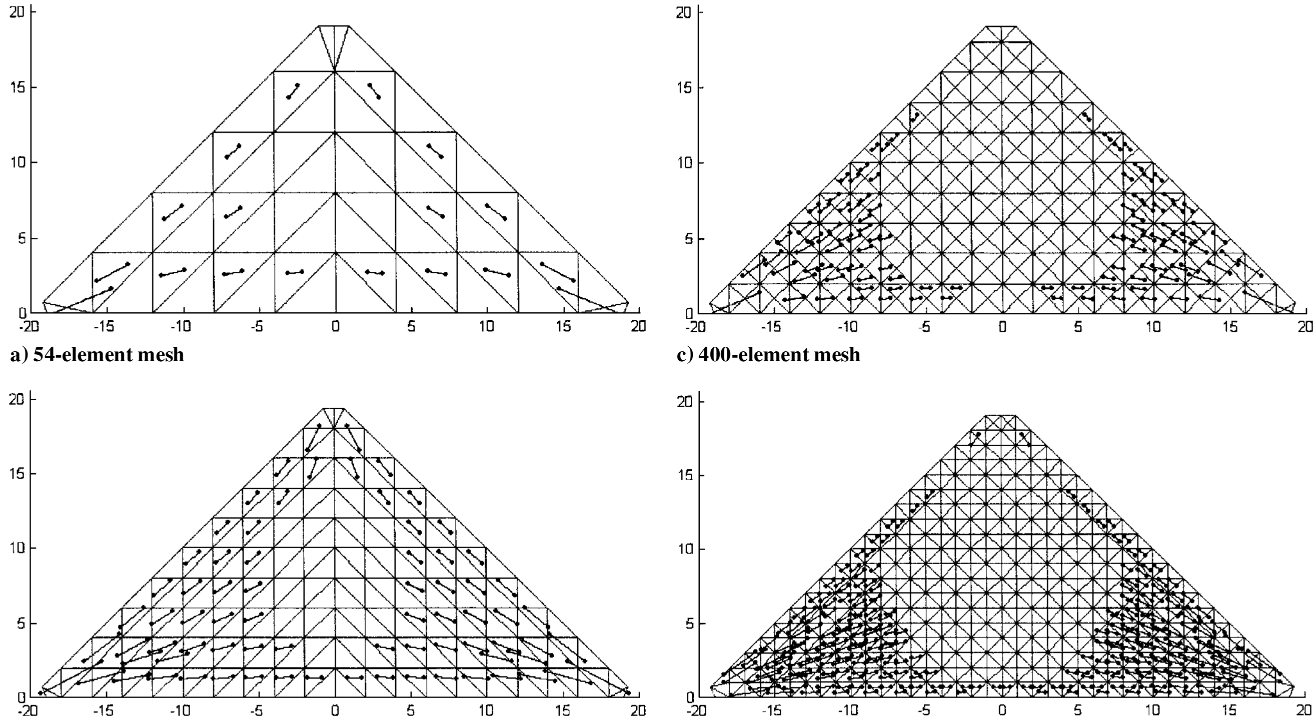


Fig. 1 Triangular membrane under tension loads.



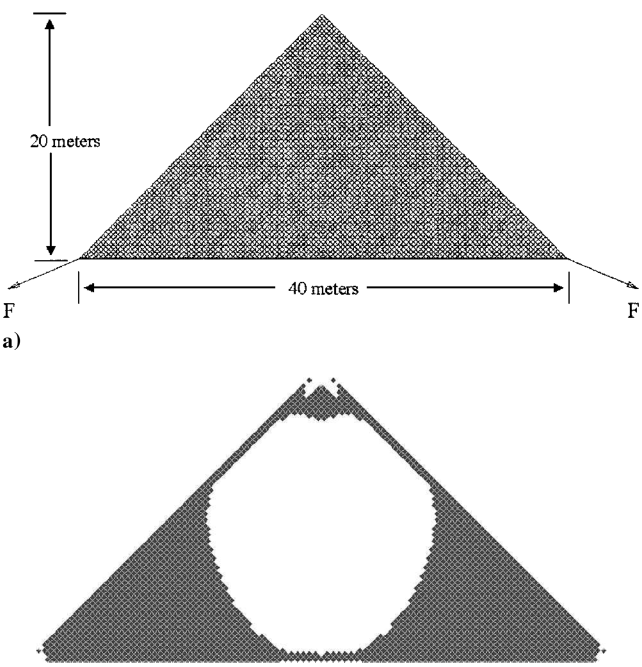
a) 54-element mesh

c) 400-element mesh

b) 204-element mesh

d) 800-element mesh

Fig. 2 Triangular membrane.



a)

b)

Fig. 3 Finite element method (FEM) results obtained by the finite element software ABAQUS: a) FEM mesh of 100 triangular elements and 4950 quadrant elements and b) wrinkled region.

the proposed method with the four different finite element meshes. The results given in Table 1 show good convergence of the stress as the number of elements increases. For this example, it is found that 400 elements are enough to assure reasonable precision of numerical solutions.

Example 2: In-Plane Bending of a Stretched Rectangular Membrane

In Fig. 4, a rectangular membrane of length L, width a, and thickness h is subjected to a pointwise force P and a moment M at the midpoint of each of the vertical sides and a uniform tension force

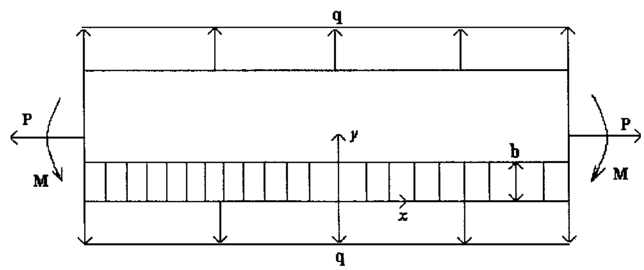


Fig. 4 In-plane bending of a stretched rectangular membrane.

q on each of the horizontal sides. This membrane was considered by Stein and Hedgepeth,[6] who obtained the following relation between the wrinkled region height b and the loads P and M:

$$2M/Pa = 1 - \tfrac{2}{3}(1 - b/a) \tag{33}$$

For convenience, dimensionless parameters are assumed: $L = 3$, $a = 1$, $h = 0.01$, $E = 100$, and $\mu = 0.3$. Also, let the uniform tension forces be fixed at $q = 2.2 \times 10^{-3}$.

Figures 5a and 5b show the wrinkled regions of the membrane predicted by the proposed method with 384 elements. Here, the combined effect of loads P and M is approximated by the horizontal nodal forces P_1; the uniform force q is approximated by the vertical nodal forces P_2. As in example 1, the dumbbell-like symbol indicates a wrinkled element. The loads in Fig. 5a are $P = 5P_1$ and $M = \frac{5}{4}P_1 a$, which indicates $2M/Pa = \frac{1}{2}$ and, by Eq. (33), gives the width of the wrinkled region $b = a/4$. The computed wrinkled region in Fig. 5a has good agreement with the analytical prediction [Eq. (33)]. The loads in Fig. 5b are $P = 3P_1$ and $M = \frac{9}{8}P_1 a$, which, by Eq. (33), gives $b = \frac{5}{8}a$. The wrinkled region obtained by the proposed method also well matches this analytical prediction. The upper horizontal boundary of the wrinkled region in both Figs. 5a and 5b is not a straight line, due to the approximation of the loads P and moment M by the nodal forces P_1. This error, however, is small and shrinks quickly as the number of elements increases.

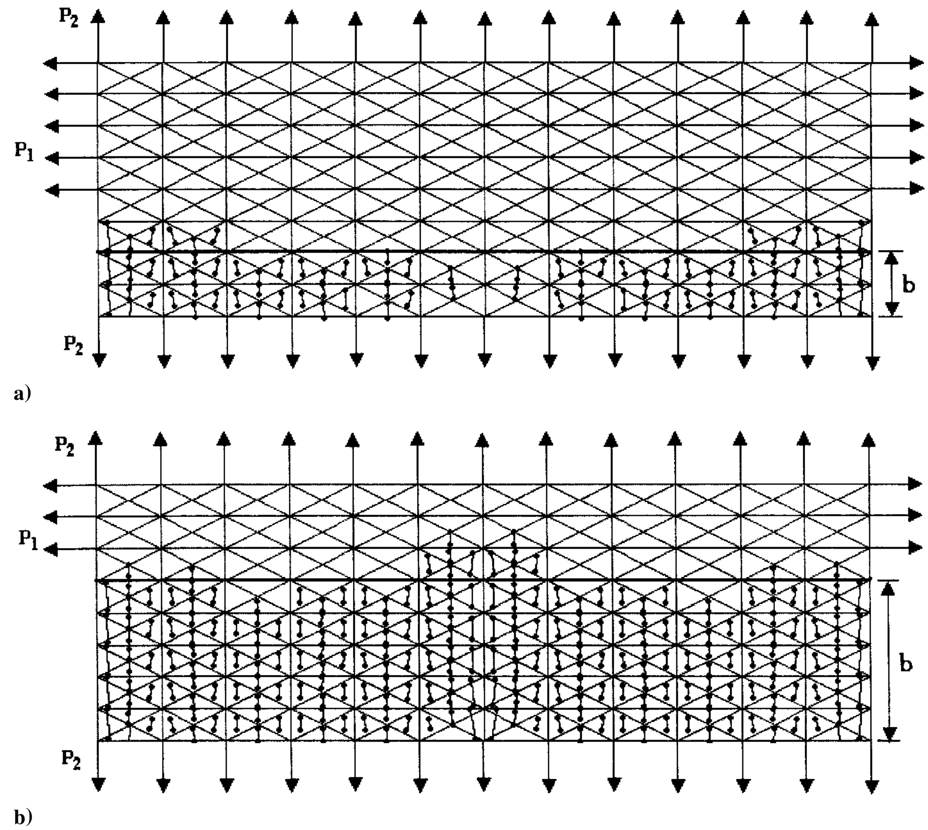


Fig. 5 Rectangular membrane: a) $P = 5P_1$ and $M = \frac{5}{4}P_1 h$ and b) $P = 3P_1$ and $M = \frac{9}{8}P_1 h$.

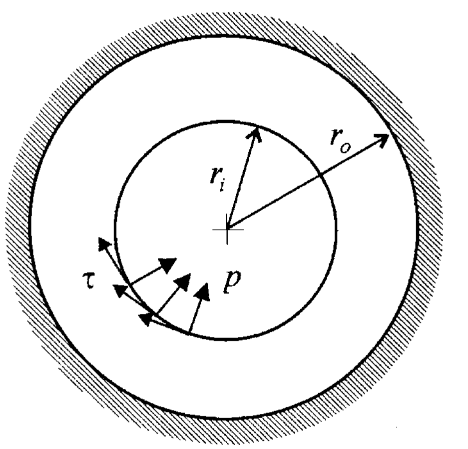


Fig. 6 Annular membrane with a fixed outer edge.

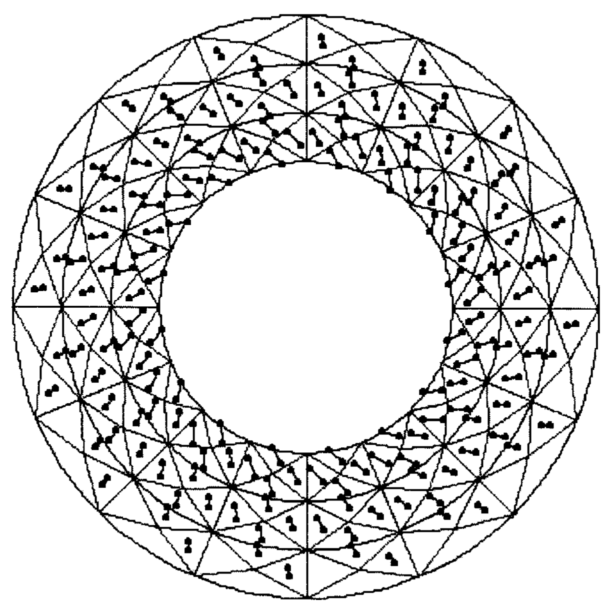


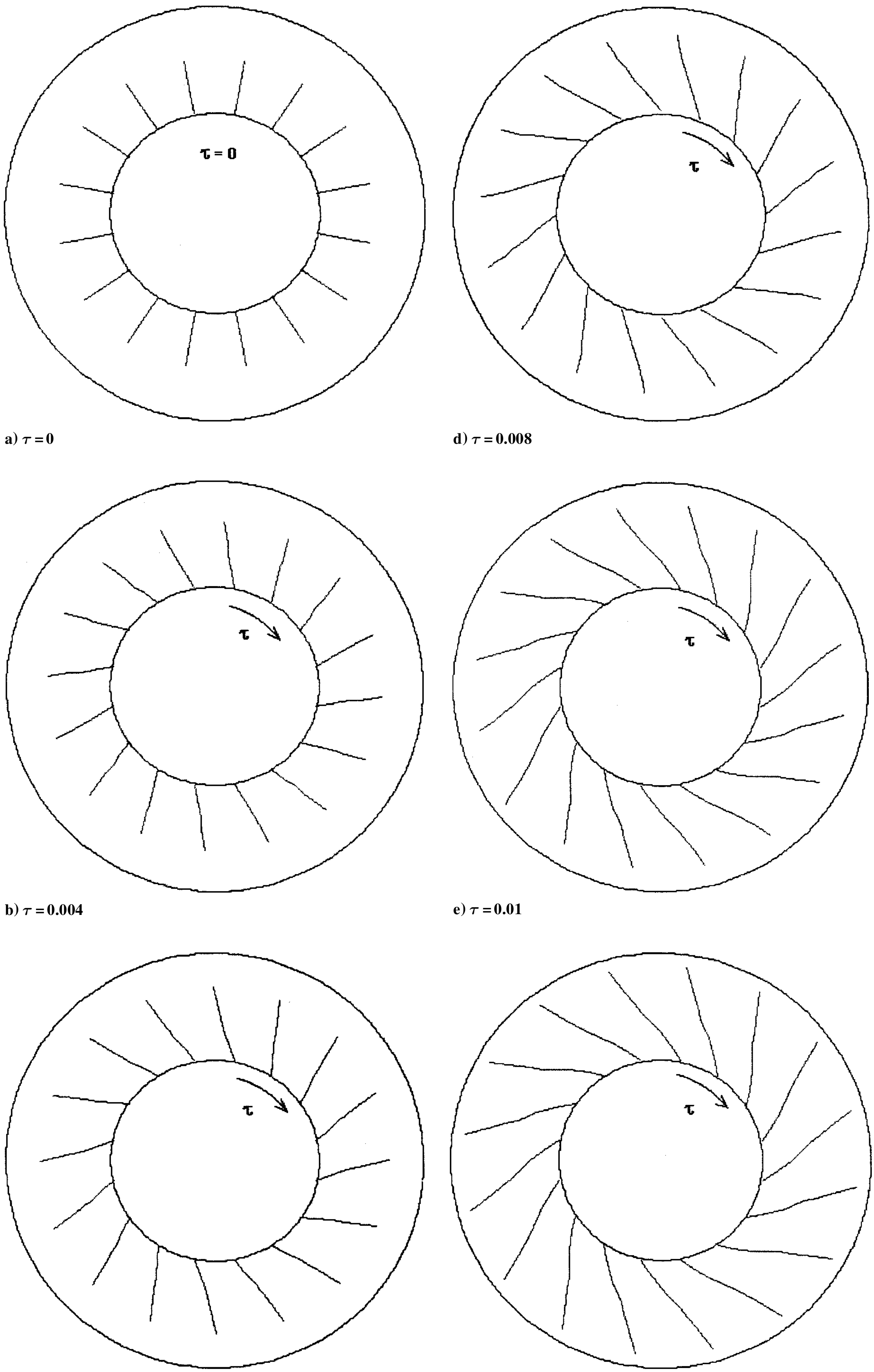
Fig. 7 Wrinkled pattern of the annular membrane, $p = 0.01$ and $\tau = 0.01$.

Example 3: Annular Membrane

An annular membrane of outer radius $r_o = 6$ and inner radius $r_i = 3$ is fixed at the outer edge (Fig. 6). The membrane is subjected to a normal force p and a tangential force τ at each of 16 equally spaced points along the inner edge. Figure 7 shows the wrinkled pattern of the membrane under the loads $p = 0.01$ and $\tau = 0.01$ predicted by the proposed method with a 192 element.

Now fix the normal force at $p = 0.01$ and change tangential load τ from 0. The evolution of the wrinkle pattern of the membrane under the changing tangential load is presented in Figs. 8a–8f. To better illustrate wrinkle patterns, a wrinkle line drawing technique has been developed, in which a wrinkle line is defined as a trajectory of the first principal stress σ_1 that continuously crosses all adjacent wrinkled elements. When the tangential force is zero, the wrinkle lines are in the radial direction, as shown in Fig. 8a. With the increase of the tangential force, wrinkle lines tilt from the radial direction toward the circumferential direction of the membrane and, at the same time, stretch out toward the outer edge of the membrane. When the tangential force $\tau = 0.0115$, most of the membrane has been wrinkled (Fig. 8f). Further numerical simulation indicates that the entire membrane is wrinkled around $\tau = 0.0117$.

It is well known that a two-dimensional wrinkled membrane with zero bending stiffness has an infinite number of wrinkles.[4,5] Therefore, a membrane model that is based on two-dimensional elasticity theory and the tension field theory can only predict the orientation of wrinkles (σ_1 lines) and the shapes of wrinkle regions.



a) $\tau = 0$

d) $\tau = 0.008$

b) $\tau = 0.004$

e) $\tau = 0.01$

c) $\tau = 0.006$

f) $\tau = 0.0115$

Fig. 8 Evolution of the wrinkled pattern of the annular membrane subject to a constant normal force ($p = 0.01$) and a changing tangential load.

VIII. Conclusions

A 1-VP membrane model with an associate numerical method has been proposed for numerical analysis of partially wrinkled model. The highlights of the work presented in this paper are summarized as follows.

1) Through introduction of a nonnegative variable parameter in Poisson's ratio, the 1-VP membrane model systematically represents the unwrinkled and wrinkled states of wrinkled membranes. The variable parameter physically characterizes strain relaxation in wrinkled regions.

2) A PVP for the 1-VP model is developed, based on which a finite element discretization is obtained for wrinkled membranes of arbitrary shapes and boundary conditions.

3) With the parametric potential energy functional, the original wrinkling problem of membranes is converted to an NCP, which can be solved by an efficient numerical algorithm such as the smoothing Newton method. One special feature of the NCP is that its solution does not require iteration of membrane stresses and, thus, avoids divergence problems commonly encountered in stress iteration.

4) The proposed membrane modal and numerical method expand the utility of the Stein and Hedgepeth theory from a few simple-shaped membranes to membranes of arbitrary shapes and boundary conditions, which have various technological applications.

The 1-VP membrane model is limited to membranes with taut and wrinkled state. Recent research shows that through introduction of a second variable parameter slack state of membranes can be included in a wrinkling analysis. This new result will be reported in a follow-up paper. Hence, introduction of variable parameters in membrane modeling provides a promising tool for efficient numerical analysis of wrinkled membranes.

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